**NUMERICAL METHODS   
FOR MATHEMATICAL PHYSICS INVERSE PROBLEMS**

## Lecture 8. Inverse problems for linear parabolic equations

We know that the inverse problems can be transformed to the problems of finding of extremum. So the practical methods of inverse problems theory are based on the optimization methods. The problems of the minimization of the functionals can be solved by means of the gradient methods. We applied it for the cases with direct dependence of the functional from unknown parameter of the system. However for the standard optimization control problems this dependence is not direct. In really the functional depends from the state function; and the state function depends from unknown parameter (control) by the state equation. It is true for optimization control problems, which are the transformation of mathematical physics inverse problems. So we will try to extend the known minimization methods to the optimization control problems.

### 8.1. Inverse problem with distributed parameter and distributed measuring

Let Ω be *n*-dimensional set with boundary *S*. Consider the system described by the heat equation

 (8.1)

with initial condition

 (8.2)

and boundary condition

 (8.3)

where the functions *ϕ* and *ψ* are known, and the function *v* is unknown. We have also the measuring condition

 (8.4)

where the function *z* is given. We would like to determine the function *v* such that the solution *y* of the boundary problem (8.1) – (8.3) satisfies the equality (8.4). This inverse problem can be transformed to the problem of the minimization of the functional



where  and  is the solution of the system (8.1) – (8.3) for the control *v.*

We will solve this problem with using gradient method. Find the derivative of the functional at a point *v*.

### 8.2. Inverse problem with distributed parameter and pointwise measuring

We consider the equation (8.1) with distributed parameter *v*, boundary conditions (8.2), (8.3) and pointwise measuring

 (8.5)

where  are given points of the set Ω (points of measuring), and  are given numbers (results of measuring). In this case the minimizing functional will be determined by the formula



### 8.3. Inverse problem with initial parameter and final measuring

Consider the system described by the heat equation

 (8.6)

with initial condition

 (8.7)

and boundary condition

 (8.8)

where the functions *ϕ* and *ψ* are known, and the function *v* is unknown. We have also the measuring condition

 (8.9)

We would like to determine the function *v* such that the solution *y* of the boundary problem (8.6) – (8.8) satisfies the equality (8.9). This inverse problem can be transformed to the problem of the minimization of the functional



### 8.4. Inverse problem with boundary parameter and boundary measuring

Let the boundary *S* of the given set Ω consists of two part *S*1 and *S*2. We consider the heat equation (8.6) with initial condition (8.7) and boundary conditions

 (8.10)

 (8.11)

where the function *g* is given, and the function *v* is unknown parameter. The additional condition is determined by the equality

 (8.12)

where *z* is given function (result of measuring), and the term at the left side of this equality is the normal derivative of the state function *y*. Thus we have two conditions (8.10) and (8.12) on the part *S*1 of the boundary. But we do not any condition on the *S*2.

Transform this inverse problem to the optimization control problem. Determine the functional



So we have the problem of the minimization of the functional *I*, where *y*[*v*] is the solution of the equation (8.6) with boundary conditions (8.7), (8.10), (8.11).

Find the derivative of the functional at the point *v*. Determine the difference

 (8.13)

where *σ* is a constant, *h* is an arbitrary function,  The function *ϕ* is the solution of the boundary problem

 (8.14)

 (8.15)

 (8.16)

Multiply the equality (8.14) to the arbitrary function *p*. After integration we obtain



Using second Green’s formula, we get



This equality can be transform to

 (6.18)

because of the boundary conditions (6.16), (6.17).

The equality (6.18) is true for all function *p*. Let *p* is the solution of the boundary problem

 (6.19)

 (6.20)

 (6.21)

Then the equality (6.18) is transformed to



So we get

 (6.22)

Estimate the second term at the right side of this equality. Multiply the equality (6.15) to the arbitrary function *p* and integrate the result with respect to *x* of the set Ω. We obtain



Using first Green’s formula, we get



because of the equalities (6.16), (6.17).

**Task 1**

Find the derivative and determine the gradient method for the following problems with unknown function *v*:

1) 

2) 

3) 

**Task 2**

Find the derivative and determine the gradient method for the following problems with unknown function *v*:

1) 

2) 

3) 